

Note

**Reduction of the Dirichlet Problem
to an Initial Value Problem***

1. INTRODUCTION

Initial-value problems are desirable for the computational solution of problems of mathematical physics. They are especially suited for modern calculators, both for the numerical simplicity of the solution methods and for the degree of precision achieved.

This paper presents an initial-value approach for the solution of one of the classical problems of mathematical physics, the Dirichlet problem. Although the derivation is concerned with solutions of this problem for plane regions with prescribed boundary values, the approach presented here could be easily generalized to higher dimensions.

The initial-value method is derived by a combination of invariant imbedding techniques and the Fredholm integral equation method of representation of the potential as a function of a dilayer distribution on the boundary of the region in question.

Section 2 presents an invariant imbedding approach to the solution of a certain family of Fredholm integral equations. Section 3 presents the Fredholm integral equation method for the solution of the Dirichlet problem. Section 4 is devoted to the statement of the initial-value problem suggested by the results of the preceding sections and to the proof of its validity. Section 5 presents the numerical approach and experiments. Section 6 discusses the advantages of the method used.

2. REDUCTION OF A FAMILY OF FREDHOLM INTEGRAL EQUATIONS

In a previous paper [1], it was proved that the family of integral equations

$$\sigma(t, \lambda) = g(t) + \lambda \int_0^1 k(t, y) \sigma(y, \lambda) dy \tag{1}$$

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and the Cauchy system

$$\sigma_\lambda(t, \lambda) = \int_0^1 K(t, y, \lambda) \sigma(y, \lambda) dy, \quad (2)$$

$$K_\lambda(t, y, \lambda) = \int_0^1 K(t, y', \lambda) K(y', y, \lambda) dy', \quad 0 \leq t, y \leq 1, \quad 0 \leq \lambda, \quad (3)$$

$$\sigma(t, 0) = g(t), \quad (4)$$

$$K(t, y, 0) = k(t, y), \quad 0 \leq t, y \leq 1, \quad (5)$$

are equivalent for values of λ such that solutions of Eq (1) exist for those λ and solutions of the Cauchy system exist and are unique in the interval $[0, 1]$ of the real line. The subscript notation in the above equations indicates partial derivation with respect to the subscript. For a quick derivation of the preceding Cauchy system please see Appendix 1.

3. THE FREDHOLM INTEGRAL EQUATION METHOD

This section presents an approach to the solution of the Dirichlet problem in the plane by reduction to a Fredholm integral equation of the second order.

The exposition here follows that of the work of Courant and Hilbert [2, p. 298] in which further details can be found.

Let R be a plane region bounded by a boundary curve Γ which will be assumed to be represented parametrically by two functions $x(t)$ and $y(t)$ which are differentiable up to the fourth order.

The Dirichlet problem for R may be stated as follows: Find a function $u(x, y)$ such that

$$\nabla^2 u(x, y) = 0 \text{ in } R, \quad (6)$$

$$u(x(t), y(t)) = f(t) \text{ on } \Gamma, \quad (7)$$

where $f(t)$ is a known function.

If the function $u(x, y)$ is represented as

$$u(x, y) = \int_\Gamma \sigma(t) \frac{\partial \gamma}{\partial \nu} dt, \quad (8)$$

where

$$\gamma = \log \frac{1}{r}, \quad (9)$$

$$r = [(x - x(t))^2 + (y - y(t))^2]^{1/2}, \quad (10)$$

and ν is the exterior unit normal to Γ at $(x(t), y(t))$, then $\sigma(t)$ satisfies a Fredholm integral equation.

The representation (8) corresponds physically to the representation of the potential $u(x, y)$ inside the region as that caused by a double layer of density $\sigma(t)$ on the boundary Γ .

The integral equation satisfied by $\sigma(t)$ is

$$\sigma(t) = -\frac{1}{\pi} f(t) + \int_{\Gamma} k(t, s) \sigma(s) ds, \quad (11)$$

where

$$k(t, s) = -\frac{1}{\pi} \frac{[\dot{y}(t)(x(t) - x(s)) - \dot{x}(t)(y(t) - y(s))]}{[(\dot{x}(t)^2 + \dot{y}(t)^2)^{1/2} ((x(t) - x(s))^2 + (y(t) - y(s))^2)]}, s \neq t, \quad (12)$$

and

$$k(t, t) = -\frac{1}{2\pi} \kappa(t), \quad (13)$$

where $\kappa(t)$ is the curvature at $(x(t), y(t))$. The curvature can be explicitly written as a function of the parameterized coordinates as

$$\kappa(t) = \frac{\dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t)}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{3/2}}. \quad (14)$$

In the preceding formulas, the convention of representing parametric derivatives of the coordinate functions by dotting the corresponding functions has been followed.

4. THE INITIAL VALUE METHOD FOR THE DIRICHLET PROBLEM

The results of the preceding section can now be used to formulate an initial value problem to solve the family of integral equations

$$\sigma(t, \lambda) = -\frac{1}{\pi} f(t) + \lambda \int_{\Gamma} k(t, s) \sigma(s, \lambda) ds, t \geq 0, \lambda \geq 0. \quad (15)$$

When the kernel $k(t, s)$ is given by Eqs. (12) and (13), then $\sigma(t, 1)$ is the solution of Eq. (11). This allows computation of the potential function $u(x, y)$ by use of the representation formula (8), where $\partial\gamma/\partial\nu$ may be written explicitly as

$$\frac{\partial\gamma}{\partial\nu} = \frac{[\dot{y}(t)(x - x(t)) - \dot{x}(t)(y - y(t))]}{[(\dot{x}(t)^2 + \dot{y}(t)^2)^{1/2} ((x - x(t))^2 + (y - y(t))^2)]} \quad (16)$$

The family of equations (15) is of the type analyzed in Section 2. Accordingly an initial-value problem may be posed as

$$\sigma_\lambda(t, \lambda) = \int_{\Gamma} K(t, s', \lambda) \sigma(s', \lambda) ds', \quad (17)$$

$$K_\lambda(t, s, \lambda) = \int_{\Gamma'} K(t, s', \lambda) K(s', s, \lambda) ds', \quad (18)$$

$$\sigma(t, 0) = -\frac{1}{\pi} f(t), \quad (19)$$

$$K(t, s, 0) = k(t, s), \quad (20)$$

where $k(t, s)$ is given by Eqs. (12) and (13).

Following Courant and Hilbert [2, p. 301] it can be proved that the family of integral equations

$$\sigma(t, \lambda) = \lambda \int_{\Gamma} k(t, s) \sigma(s, \lambda) ds, \quad \lambda \geq 0, \quad (21)$$

does not have an eigenvalue in the interval $[0, 1]$ of the real line. This implies that the solution of Eq. (19) is identically zero for values of λ in the $[0, 1]$ interval. Fredholm's theory assures then the existence of a solution for the nonhomogeneous equation (15) for those values of λ and for any function f continuously differentiable. It was stated previously that existence of a unique solution of the initial-value problem and existence of the solution of Eq. (15) suffice for the equivalence of both problems. This implies that existence of a unique solution of the Cauchy system suffices for equivalence of both problems for any value of λ less than or equal to one.

5. NUMERICAL TREATMENT OF THE INITIAL VALUE PROBLEM

To solve numerically the system given by Eqs. (17)–(20), the integrals on the right side of Eqs. (17) and (18) were approximated by use of the trapezoidal integration formula. This formula was selected because of the particular nature of the integrands which are continuous functions defined on a closed boundary [3].

If $x(t), y(t), 0 \leq t \leq T$, are the parametric coordinates of the boundary Γ and m is the number of intervals into which the boundary is to be discretized, then the approximating system may be written as

$$\frac{d}{d\lambda} \sigma_i(\lambda) = \frac{T}{m} \sum_{j=1}^m K_{ij}(\lambda) \sigma_j(\lambda), \quad (22)$$

$$\frac{d}{d\lambda} K_{i\ell}(\lambda) = \frac{T}{m} \sum_{j=1}^m K_{ij}(\lambda) K_{j\ell}(\lambda), \quad (23)$$

$$\sigma_i(0) = -\frac{1}{\pi} f(t_i), \quad (24)$$

$$K_{i\ell}(0) = k(t_i, t_\ell) \quad (25)$$

$$t_i = (i-1) \frac{T}{m}. \quad (26)$$

In the preceding system, $\sigma_i(\lambda)$ is the approximant of $\sigma(t_i, \lambda)$ and $K_{i\ell}(\lambda)$ is the approximant of $K(t_i, t_\ell, \lambda)$. The discrete system contains $m^2 + m$ equations. Solution of the system given by Eqs. (22)–(26) yields values $\sigma_i(1), 1 \leq i \leq m$, which may be used in a trapezoidal approximation of the representation formula (8) to compute values of the potential in the interior of the region under consideration.

Integration of the discretized system was performed in an IBM 360/91 system using a fourth order Runge–Kutta scheme. The step used for λ was $h = 0.02$.

In the first example the region was a unit circle discretized into 50 equal intervals. The boundary condition was selected as

$$f(t) = \cos^2 t, \quad 0 \leq t \leq 2\pi. \quad (27)$$

The exact solution for this case can be exactly expressed as

$$\sigma(t, 1) = -\frac{1}{\pi} (\cos^2 t - 0.25). \quad (28)$$

Computation of the approximate solution using the just-outlined scheme yields an accuracy superior to five significant figures.

The second example was the computation of the potential inside the rectangular region $0 \leq x \leq 12, 0 \leq y \leq 8$ with the boundary conditions

$$f = x(12 - x), \quad (29)$$

on the sides parallel to the x axis, and

$$f = -y(8 - y) \quad (30)$$

on the other sides.

The boundary was discretized in 40 intervals. This particular boundary does not satisfy the differentiability requirements at the boundary stated in Section 3. However, the potential in the region can be arbitrarily approximated by the potential in smooth boundary regions sufficiently close to it. Numerically, two possible courses are open: first, to select the partition points different from any "corners"; second, to define the value of the kernel by side continuity or similar conventions. The latter alternative was chosen by defining the values of the parametric derivatives of the boundaries as the average of the side limits. The computed solution shown in Table I had an accuracy of 5 significant figures for points near to the center of the region. The solution near the boundaries is not so precise, due to the error introduced at the corners. The exact solution is

$$u(x, y) = x(12 - x) - y(8 - y). \quad (31)$$

Computing times were 99 seconds for the first example and 58 seconds for the second. They could be greatly reduced by the use of coarser grids and other quadrature formulas.

6. DISCUSSION

An initial value problem equivalent to a Dirichlet problem in the plane has been presented. Solution of the initial value problem is a feasible task for modern digital computers. Besides the high accuracy of the procedure, several other advantages are worth mentioning.

The integration procedure depends only on the particular region and boundary used for the definition of the initial values (Eqs. (19) and (20)) of the Cauchy system. This reduces sensitivity of the computation time to the particular problem. The integration time is dependent only on the integration method used and on the number of discrete partitions used in the boundary. The generality of the method with respect to region shapes is obvious. Although, depending on the integration method used, considerable storage is needed during the integration of the Cauchy system, storage of a unidimensional function, $\sigma(t, \lambda)$, is sufficient to compute the potential at any point inside the region. This compares favorably to storage of a two-dimensional function as required by conventional difference methods.

The procedure outlined here may be used for an open curve Γ . In this case, the potential in the plane generated by a dilayer at Γ , when the potential on the exterior (as defined by the normal to Γ) is prescribed, can be computed.

The derivation outlined here could be generalized to higher dimensions, and other types of potential problems can be subjected to a similar treatment.

TABLE I
 Computed Values of the Potential for the Rectangular Region $0 \leq x \leq 12, 0 \leq y \leq 8$.

$x \backslash y$	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0
0.0	0.000	11.000	20.000	27.000	32.000	35.000	36.000	35.000	32.000	27.000	20.000	11.000	0.000
1.0	-7.000	3.9095	13.069	20.106	25.126	28.137	29.141	28.137	25.126	20.106	13.069	3.9095	-7.000
2.0	-12.000	-1.1250	7.9792	14.987	19.991	22.993	23.994	22.993	19.991	14.987	7.9792	-1.1250	-12.000
3.0	-15.000	-4.1240	4.9854	11.987	16.990	19.991	20.992	19.991	16.990	11.987	4.9854	-4.1240	-15.000
4.0	-16.000	-5.1255	3.9876	10.988	15.990	18.991	19.992	18.991	15.990	10.988	3.9876	-5.1255	-16.000
5.0	-15.000	-4.1240	4.9854	11.987	16.990	19.991	20.992	19.991	16.990	11.987	4.9854	-4.1240	-15.000
6.0	-12.000	-1.1250	7.9793	14.987	19.991	22.993	23.994	22.993	19.991	14.987	7.9793	-1.1250	-12.000
7.0	-7.000	3.9095	13.069	20.106	25.126	28.137	29.141	28.137	25.126	20.106	13.069	3.9095	-7.000
8.0	0.000	11.000	20.000	27.000	32.000	35.000	36.000	35.000	32.000	27.000	20.000	11.000	0.000

APPENDIX: DERIVATION OF THE CAUCHY SYSTEM

The following is a derivation of the Cauchy system (Eqs. (2)–(5)) introduced in Section 2. This derivation is given mainly as a plausibility argument for such a system.

Let $u(t, \lambda)$ be defined as the solution of the integral equation

$$u(t, \lambda) = g(t) + \lambda \int_0^1 k(t, y) u(y, \lambda) dy. \quad (\text{a-1})$$

Differentiation of both sides of this equation with respect to λ yields

$$u_\lambda(t, \lambda) = \int_0^1 k(t, y) u(y, \lambda) dy + \lambda \int_0^1 k(t, y) u_\lambda(y, \lambda) dy. \quad (\text{a-2})$$

Let $K(t, y, \lambda)$ be introduced as the solution of the integral equation

$$K(t, y, \lambda) = k(t, y) + \lambda \int_0^1 k(t, y') K(y', y, \lambda) dy'. \quad (\text{a-3})$$

Comparison of equations (a-2) and (a-3) and use of the superposition principle for linear systems yields

$$u_\lambda(t, \lambda) = \int_0^1 K(t, y, \lambda) u(y, \lambda) dy, \quad (\text{a-4})$$

thus establishing Eq. (2), Section 2.

Differentiation of both sides of Eq. (a-3), on the other hand, yields

$$K_\lambda(t, y, \lambda) = \int_0^1 k(t, y') K(y', y, \lambda) dy' + \lambda \int_0^1 k(t, y') K_\lambda(y', y, \lambda) dy', \quad (\text{a-5})$$

and comparison with Eq. (a-3) and use of the superposition principle yields

$$K_\lambda(t, y, \lambda) = \int_0^1 K(t, y', \lambda) K(y', y, \lambda) dy', \quad (\text{a-6})$$

which is Eq. (3), Section 2.

Equations (4) and (5), Section 2 are readily established from Eq. (a-1) and (a-3) by setting $\lambda = 0$.

Equation (a-3), in particular, is the differential form of a wellknown formula for resolvents of integral equations¹

$$K(t, y, \lambda) - K(t, y, \lambda') = (\lambda - \lambda') \int_0^1 K(t, y', \lambda) K(y', y, \lambda') dy'. \quad (\text{a-7})$$

¹ Courant, R. and Hilbert, D. "Methods of Mathematical Physics," Vol. 1 (p. 141, Eq. (63)), Interscience, New York, 7th edition, 1966.

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